

BOHNENBLUST–HILLE INEQUALITY FOR POLYNOMIALS WHOSE MONOMIALS HAVE UNIFORMLY BOUNDED NUMBER OF VARIABLES

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ABSTRACT. In 2015, using an innovative technique, Carando, Defant and Sevilla-Peris succeeded in proving a Bohnenblust–Hille type inequality with constants of polynomial growth in m for a certain family of complex m -homogeneous polynomials. In the present paper, using a completely different approach, we prove that the constants of this inequality are uniformly bounded irrespectively of the value of m .

1. INTRODUCTION

The Bohnenblust–Hille inequality [3] for complex m -homogeneous polynomials asserts that there is a constant $C_m > 0$ such that

$$\left(\sum_{|\alpha|=m} |c_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|P\|$$

for all continuous m -homogeneous polynomials $P : c_0 \rightarrow \mathbb{C}$ of the form $P(x) = \sum_{|\alpha|=m} c_\alpha(P) \mathbf{x}^\alpha$. This inequality is important in many fields of Mathematics. In 2011, it has been proven in [5] that C_m can be chosen with exponential growth, and this result had several important applications. In 2014 the estimates of [5] were improved in [2] and it has been shown that for any $\varepsilon > 0$ there is a constant $\kappa > 0$ such that

$$C_m \leq \kappa (1 + \varepsilon)^m$$

and this result was crucial to obtain the final solution to the asymptotic growth of the Bohr radius problem. In 2015, Carando, Defant and Sevilla-Peris have shown that for a particular family of m -homogeneous polynomials the constants C_m could have been chosen to have polynomial growth in m . More precisely, they have proved that for polynomials whose monomials have a uniformly bounded number M of different variables, there is a Bohnenblust–Hille type inequality with a constant of polynomial growth in m . Our main result shows, by means of a completely different technique, that in fact these constants are uniformly bounded by a constant that does not depend on m .

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , let $\alpha = (\alpha_j)_{j=1}^\infty$ be a sequence in $\mathbb{N} \cup \{0\}$ and, as usual, define $|\alpha| = \sum \alpha_j$; in this case we also denote $\mathbf{x}^\alpha := \prod_j x_j^{\alpha_j}$. An m -homogeneous polynomial $P : c_0 \rightarrow \mathbb{C}$ is denoted by

$$P(x) = \sum_{|\alpha|=m} c_\alpha(P) \mathbf{x}^\alpha.$$

We recall that the norm of P is given by $\|P\| := \sup_{x \in B_{c_0}} |P(x)|$. Since $|\alpha| = m$, it is obvious that only a finite number of α_j are non null and we define $\binom{m}{\alpha} := \frac{m!}{\alpha_1! \dots \alpha_n!}$, where $\alpha_1, \dots, \alpha_n$ are the non null elements of α . Following the notation of [4], for positive integers m and $M \leq m$ we define

$$\text{vars}(P) = \text{card} \{j : \alpha_j \neq 0\}$$

and

$$\Lambda_M = \{\alpha : |\alpha| = m, \text{ vars}(P) \leq M\}.$$

Key words and phrases. Bohnenblust–Hille inequality.

M. Maia and T. Nogueira are supported by Capes and D. Pellegrino is supported by CNPq.

In [4] it was proved that

$$(1) \quad \left(\sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq 2^{\frac{M}{2}} m^{\frac{M+1}{2}} \|P\|$$

for all continuous m -homogeneous polynomials $P : c_0 \rightarrow \mathbb{C}$, being stressed by the authors of [4] the polynomial growth (in m) of the constants – this was a very nice property having in mind that in general the best known estimates had just a subexponential growth (in m). Our main result shows that the optimal constants of (1) are universally bounded irrespectively of the value of m :

Theorem 1.1. *For all positive integers m and $M \leq m$, there exists a constant $\kappa_M > 0$ such that*

$$\left(\sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq \kappa_M \|P\|$$

for all continuous m -homogeneous polynomials $P : c_0 \rightarrow \mathbb{C}$.

2. THE PROOF OF THEOREM 1.1

The main result of [1] asserts that if $1 \leq k \leq m$ and $n_1, \dots, n_k \geq 1$ are positive integers such that $n_1 + \dots + n_k = m$, then there is a constant $C_{k,m}^{\mathbb{K}} \geq 1$ such that

$$(2) \quad \left(\sum_{i_1, \dots, i_k=1}^{\infty} \left| T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k}) \right|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \leq C_{k,m}^{\mathbb{K}} \|T\|$$

for all continuous m -linear forms $T : c_0 \times \dots \times c_0 \rightarrow \mathbb{K}$. Moreover, the exponent $\frac{2k}{k+1}$ is optimal. In [1] it is also proved that

$$(3) \quad C_{k,m}^{\mathbb{K}} \leq C_k^{\mathbb{K}}$$

for all $1 \leq k \leq m$, where $C_k^{\mathbb{K}}$ is the optimal constant of the k -linear Bohnenblust–Hille inequality. From [2] we know that there are positive constants β_1, β_2 such that

$$(4) \quad C_k^{\mathbb{C}} \leq \beta_1 k^{\frac{1-\gamma}{2}} < \beta_1 k^{0.212}$$

and

$$C_k^{\mathbb{R}} \leq \beta_2 k^{\frac{2-\log 2-\gamma}{2}} < \beta_2 k^{0.365},$$

where γ is the Euler-Mascheroni constant. From now on $C_k^{\mathbb{K}}$ will be just denoted by C_k .

Let \hat{P} be the symmetric m -linear form associated to P . Let

$$\Gamma_m := \{ \tau = (\tau_1, \dots, \tau_M) \in [0, m]^M : \tau_1 + \dots + \tau_M = m \}.$$

It is simple to verify that (an argument of symmetry provides even sharper estimates but, surprisingly, this rough estimate is enough for our purposes)

$$\sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^{\frac{2M}{M+1}} \leq \sum_{\tau \in \Gamma_m} \sum_{i_1, \dots, i_M} \binom{m}{\tau}^{\frac{2M}{M+1}} |\hat{P}(e_{i_1}^{\tau_1}, \dots, e_{i_M}^{\tau_M})|^{\frac{2M}{M+1}}.$$

If $\lfloor x \rfloor := \max\{n \in \mathbb{N} : n \leq x\}$ it is a simple exercise to verify that

$$(5) \quad \binom{m}{\tau} \leq \frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M}.$$

By (5), since

$$\sum_{i_1, \dots, i_k} |\hat{P}(e_{i_1}^{\tau_1}, \dots, e_{i_k}^{\tau_k})|^{\frac{2M}{M+1}} \leq \sum_{i_1, \dots, i_M} |\hat{P}(e_{i_1}^{\tau_1}, \dots, e_{i_M}^{\tau_M})|^{\frac{2M}{M+1}}$$

for all $1 \leq k \leq M$, we have

$$\begin{aligned} \sum_{i_1, \dots, i_M} \binom{m}{\tau}^{\frac{2M}{M+1}} |\hat{P}(e_{i_1}^{\tau_1}, \dots, e_{i_M}^{\tau_M})|^{\frac{2M}{M+1}} &\leq \left(\frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M} \right)^{\frac{2M}{M+1}} \sum_{i_1, \dots, i_M} |\hat{P}(e_{i_1}^{\tau_1}, \dots, e_{i_M}^{\tau_M})|^{\frac{2M}{M+1}} \\ &\leq \left(\frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M} \right)^{\frac{2M}{M+1}} (C_M \|\hat{P}\|)^{\frac{2M}{M+1}}, \end{aligned}$$

where in the last inequality we have used (2) and (3). Note also that

$$\begin{aligned} \sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^{\frac{2M}{M+1}} &\leq \sum_{\tau \in \Gamma_M} \left(\frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M} \right)^{\frac{2M}{M+1}} (C_M \|\hat{P}\|)^{\frac{2M}{M+1}} \\ &= \binom{m+M-1}{m} \left(\frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M} \right)^{\frac{2M}{M+1}} (C_M \|\hat{P}\|)^{\frac{2M}{M+1}}. \end{aligned}$$

Hence

$$\begin{aligned} \left(\sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} &\leq \binom{m+M-1}{m}^{\frac{M+1}{2M}} \frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M} C_M \|\hat{P}\| \\ &\leq \binom{m+M-1}{m}^{\frac{M+1}{2M}} \frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M} C_M e^m \|P\|. \end{aligned}$$

The above estimate holds for complex and real scalars. Since now we are just dealing with complex scalars, by the Maximum Modulus Principle, we have

$$\left(\sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{|\alpha|=m} |c_\alpha(P)|^2 \right)^{\frac{1}{2}} \leq \|P\|.$$

Since

$$\frac{1}{\frac{2m}{m+1}} = \frac{\theta}{\frac{2M}{M+1}} + \frac{1-\theta}{2}$$

with

$$\theta = \frac{M}{m},$$

by a corollary of the Hölder inequality, we have

$$\begin{aligned} (6) \quad \left(\sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} &\leq \left[\left(\sum_{\alpha \in \Lambda_M} |a_\alpha|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \right]^{\frac{M}{m}} \left[\left(\sum_{\alpha \in \Lambda_M} |a_\alpha|^2 \right)^{\frac{1}{2}} \right]^{1-\frac{M}{m}} \\ &\leq \left(\binom{m+M-1}{m}^{\frac{M+1}{2M}} \frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M} C_M e^m \|P\| \right)^{\frac{M}{m}} \|P\|^{1-\frac{M}{m}} \end{aligned}$$

Using the Stirling Formula we can prove that

$$(7) \quad \lim_{m \rightarrow \infty} \left[\frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M} \right]^{\frac{M}{m}} = M^M.$$

By (7) and (6) we conclude that there is a constant $\zeta_M > 0$ such that

$$\begin{aligned} \left(\sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} &\leq \binom{m+M-1}{m}^{\frac{M+1}{2m}} \left[\frac{m!}{(\lfloor \frac{m}{M} \rfloor!)^M} \right]^{\frac{M}{m}} C_M^{\frac{M}{m}} e^M \|P\| \\ &\leq \zeta_M \binom{m+M-1}{m}^{\frac{M+1}{2m}} C_M^{\frac{M}{m}} e^M \|P\| \end{aligned}$$

Recalling that γ denotes the Euler–Mascheroni constant, by the previous inequality combined with (4) we have

$$\begin{aligned} \left(\sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} &\leq \zeta_M \binom{m+M-1}{m}^{\frac{M+1}{2m}} \left(\beta_1 M^{\frac{1-\gamma}{2}} \right)^{\frac{M}{m}} e^M \|P\| \\ &\leq \kappa_M \|P\| \end{aligned}$$

for a certain $\kappa_M > 0$ and this concludes the proof.

Remark 2.1. By the above proof we also conclude that for this Bohnenblust–Hille type inequality the optimal exponent is not $\frac{2m}{m+1}$, contrary to what happens for the classical Bohnenblust–Hille inequality. We can see that the inequality holds for the smaller exponent $\frac{2M}{M+1}$ and using the optimality of the exponent $\frac{2M}{M+1}$ in [1] it seems to be just an exercise to show that this exponent $\frac{2M}{M+1}$ is sharp in this case.

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